COHERENT EXCITATION OF NONLINEAR OSCILLATORS

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ABSTRACT

Coherently driven, dissipative nonlinear oscillators, (driving kept permanently in phase with the oscillators response) are proposed as systems with interesting dynamics. Results for simple, preliminary examples, which do not show chaotic behavior, are briefly discussed.

1. INTRODUCTION

Multiple nonlinear oscillators have been shown to exhibit chaos when excited in a sinusoidal manner, the phase of the excitation being fixed, with no regard for the oscillator response. Because of the nonlinearity, however, complete results for all values of driving intensity and frequency do not add to give a response to a general type of excitation.

We consider here a particular class of problems: we assume that driving is kept in phase with the oscillations. This could represent situations naturally occurring in many systems with feedback mechanisms. Further, the ensuing dynamics might be of interest in itself. We first recall the case of a parametrically pumped planar pendulum. Later we analyse a weakly forced spherical pendulum with linear drag.

2. AN OPTIMALLY PUMPED PENDULUM

A visitor to the Cathedral of Santiago de Compostela, in Galicia, northwest Spain, can witness a peculiar medieval rite on special liturgical services[1]. A giant Censer, called Botafumeiro, is hung with a rope from a structure above the crossing, and then pushed slightly off the vertical. As it sways like a pendulum, (usually eight) men on the ground at the other end of the rope cyclically pull it and let it go at the lowest and highest points respectively. In this manner the men amplify the oscillations in small steps, finally getting the Censer up the 21 meters vault, next down in a 65 meters arch along the transept, almost grazing the ground at 42 miles per hour.

This is a clear example[2,3] of parametric amplification with optimal pumping, elsewhere analysed in complete detail[4]. Here we just consider its simplest, generic features. Both high-Reynolds quadratic drag and pumping will be considered weak, the rope massless, and the Censer a point mass.

The angular equation takes the form

$$\ddot{\theta} + \omega_o^2 \sin\theta = \varepsilon(2\dot{h}\dot{\theta} - \omega_o^2 h \sin\theta - \gamma|\dot{\theta}|\dot{\theta}) \tag{1}$$

where γ is of order unity, $\omega_o^2 \equiv g/R$, the rope length is written as $r = R(1-\varepsilon h)$, $\varepsilon \equiv \Delta R/2R$ is small, and h is varied between -1 and +1 every pumping cycle (half an oscillation period). For time variations of order ω_o^{-1}, and dropping the left-hand side of (1), one gets the known results

$$\frac{\dot{\theta}^2}{2} + \omega_o^2(1-\cos\theta) = \omega_o^2(1-\cos\theta_M) \equiv E \tag{2}$$

$$\sin\frac{\theta}{2} = \left[\sin\frac{\theta_M}{2}\right] \times \mathrm{sn}\left[\omega_o(t-t_o),\ \sin\frac{\theta_M}{2}\right]. \tag{3}$$

The amplitude θ_M, and the energy E vary slowly in a scale $\varepsilon\omega_o t = O(1)$. Averaging over a pumping cycle, say that one with $\dot{\theta} > 0$, we get

$$\frac{dE(\theta_M)}{d(\varepsilon t)} = \frac{2}{T(\theta_M)} \int_{-\theta_M}^{\theta_M} (3\omega_o^2 h \sin\theta - \gamma\dot{\theta}^2)\, d\theta \tag{4}$$

where $T \approx 4\omega_o^{-1} K\left[\sin\frac{\theta_M}{2}\right]$ is the period of the sine-amplitude elliptic function, and we use conditions $\ddot{\theta} \approx -\omega_o^2 \sin\theta$ and $\left[\text{average of } \frac{d}{dt}(h\dot{\theta}^2)\right] \approx 0$.

Equation (4) yields the slow evolution of θ_M, once h is given and (3) is used.

The energy change in a cycle is $\frac{1}{2}$ TdE/dt. Thus, Eq. (4) also gives a onedimensional return map for the angle (θ_M) at times such that $\dot{\theta} = 0$. Let θ_n be θ_M at the end of the n^{th} cycle. Then we have

$$\cos\theta_{n+1} = \cos\theta_n - 3\varepsilon\beta(\theta_n) + 4\varepsilon\gamma(\sin\theta_n - \theta_n\cos\theta_n) \tag{5}$$

where

$$\beta \equiv \int_{-\theta_M}^{\theta_M} h\sin\theta\, d\theta .$$

There is a fixed point at $\theta_n = 0$. For small θ_n we have

$$\theta_{n+1} \simeq \left(1+3\varepsilon\,\frac{\beta}{\theta_n^{\,2}}\right)\theta_n - \frac{4}{3}\,\varepsilon\gamma\theta_n^2 ,$$

that is, a transcritical saddle-node bifurcation occurs if β goes from negatible to positive, the *origin going from* stable to unstable. A stable second fixed point then appears, given by

$$3\beta(\theta_n) = 4\gamma(\sin\theta_n - \theta_n\cos\theta_n).$$

Optimal pumping corresponds to $h = \pm 1$ for $\theta \lessgtr 0$, yielding $\beta = 2(1-\cos\theta_n)$. At low θ_n, Eq. (1) is a Hill equation, and for $\varepsilon \to 0$ $\theta \simeq \theta_n \sin\omega_o t$; for $h = \sin 2\omega_o t$ $(-\pi/2 < \omega_o t < \pi/2)$, when (1) *is basically a* Mathieu equation, we get $\beta = \frac{\pi}{4}\,\theta_n^2$, as compared with an optimal value $\beta \simeq \theta_n^2$. Actually, determining the optimal cycle, for given length variation, $R \pm \frac{1}{2}\Delta R$ ($\Delta R/R$ small enough), is equivalent to finding, among all functions of given period and amplitude, that one having the largest 1^{st} Fourier coefficient. The solution is the (step) function that jumps discontinuosly between maximum and minimum values; its 1^{st} Fourier coefficient is $4/\pi$ times the size of the step.

Figure 1 shows the map corresponding to (5) for the optimal pumping of the Censer, taking into account all sorts of 2^{nd} order effects (here $6\varepsilon \simeq 0.41$, $4\varepsilon\gamma \simeq 0.51$, the rope-to-Censer mass ratio is 0.28, and a characteristic Censer length is about 0.06R). The fixed point, and both number of cycles and time required to reach reasonably close to it, all are in good agreement with observations and measurements[4).

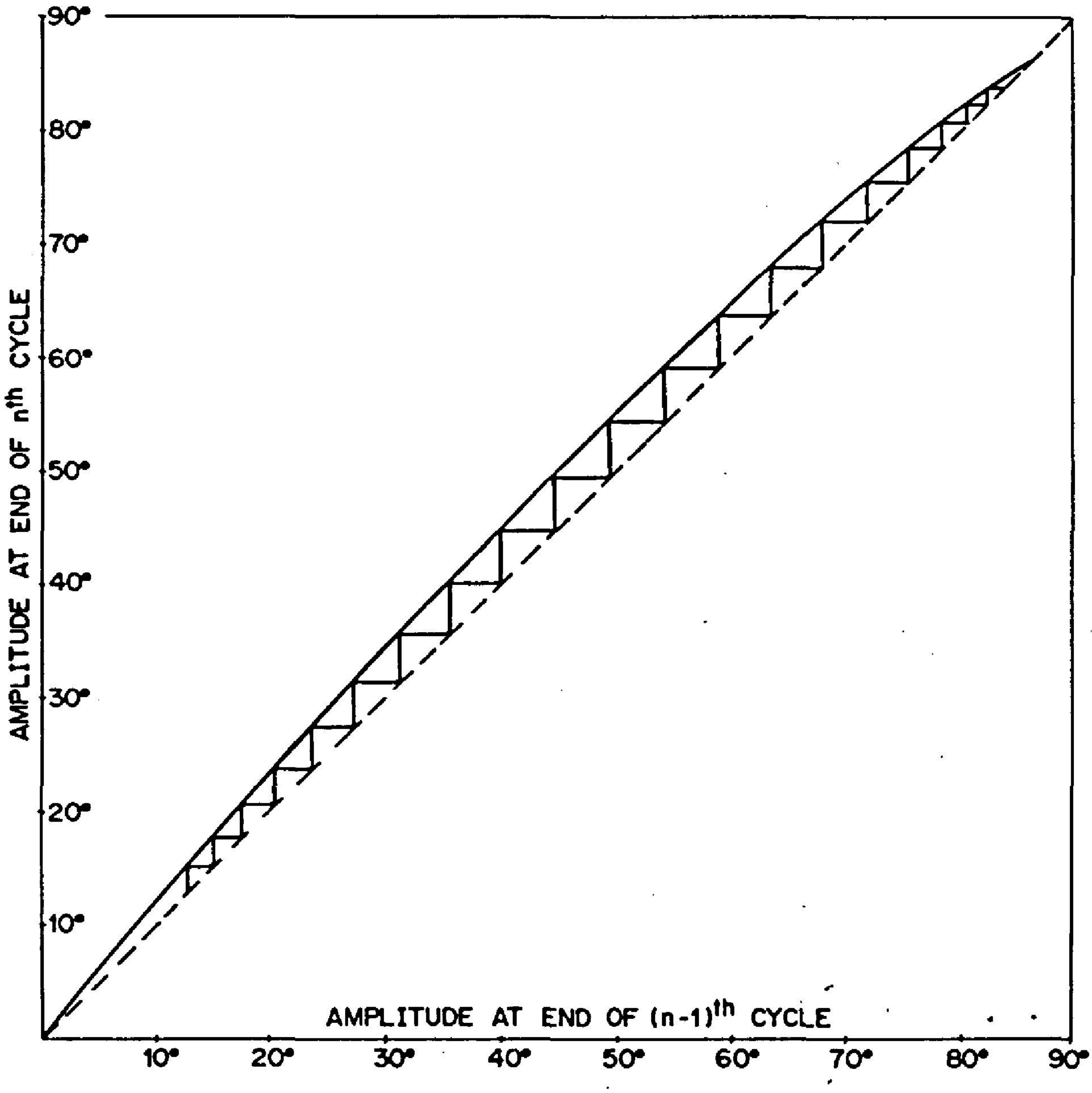

Fig. 1 Return map for the Botafumeiro, and amplitude evolution for an initial value of 13°.

3. COHERENT EXCITATION OF NON-LINEAR OSCILLATORS

Among the extensive literature on chaos in open, dissipative nonlinear dynamical systems, three basic types of models appear recurrently: i) Lorenz or Lorenz-like autonomous equations, as simple descriptions of quite complex phenomena; ii) ad-hoc iterated maps, whether two-dimensional like the Henon map, or one-dimensional and non invertible: e.g. the logistic one; and iii) sinusoidally excited, simple, second order oscillators.

Within the last type, chaos has been found for instance in such classicals as the Van der Pol[5)], Duffing[6)], and pendulum equations[7)], usually under forcing but also under parametric excitation. Since the time appears explicitly in the excitation, the system is autonomous only if considered of third order, time being the third phase coordinate (allowing chaos). The nonlinearity of the oscillator keeps its response continuously getting out of phase with the excitation. Further, the nonlinearity requires a complete exploration of excitation regimes (intensity and frequency) in order to understand the possible variety of solutions. But even then, one ends knowing quite well the response to <u>very particular</u> (purely sinusoidal) excitations.

We consider here what appears to be a natural, yet simple, problem: Nonlinear, dissipative oscillators, to which energy enters from outside in a coherent way, excitation and response kept permanently in some definite phase relationship. This would clearly require feedback mechanisms; examples of conscious "mechanisms" are, of course, the swing, -and the Censer of Sec. 2-.

A consequence of a definite phase relationship is that a 2^{nd}-order oscillator will be autonomous as second order, precluding chaos. In fact, if dissipation and energy input are both weak, then the averaging method will lead to a first-order equation [e.g. Eq. (4) of Sec. 2], phase being unimportant. Consider, however, a dissipative, dynamical system described by two or more coupled oscillators, with feedback excitation (whether forcing or pumping) such as to keep it in phase with the response of one of them. The other oscillators will keep geting out of phase. In the simplest case, a two time-scale method will lead to a 3^{rd}-order autonomous system of nonlinear differential equations, involving two oscillator amplitudes and the phase difference between them -the phase of each one by itself being again unimportant-.

An obvious candidate to the phenomenon would be the spherical pendulum. The Botafumeiro, while not quite spherical, may have been the first example of observation of chaos in simple dynamical systems: Two loose, vertical guides in the structure supporting its rope must have been introduced at some point in time clearly to prevent the development of nonplanar (irregular?) motion. Short, or moderately short, Foucault pendulums with Charron rings, which manifest irregular motion quite easily, might also be candidates to chaos[8)].

4. THE WEAKLY NONLINEAR SPHERICAL PENDULUM

We consider here a simple, preliminary problem: the weakly nonlinear motion of a spherical pendulum of radius R and small linear drag, under light, coherent forcing due, say, to the horizontal displacement of its point of support. The response of the pendulum would be simple harmonic motion at the forcing frequency ω, along both (x,y) horizontal axes, if nonlinearity and drag are neglected; taking these effects into account will produce a slow evolution of amplitudes and phases.

In the usual analysis[9], the driving force (per unit mass), along the x-axis say, would be $f_x = \varepsilon g \sin\omega t$, $\varepsilon \ll 1$, its phase being independent of the response. Introduce then $\nu \equiv (\omega_o^2-\omega^2)/\omega_o^2\,\varepsilon^{2/3}$, $x \equiv \varepsilon^{1/3}R\,a\cos(\omega t+\theta)$, $y \equiv \varepsilon^{1/3}R\,b\cos(\omega t+\psi)$, $\alpha \equiv 2\delta/\varepsilon^{2/3}$ and $\tau \equiv \frac{1}{2}\,\varepsilon^{2/3}\omega_o t$, where the drag per unit mass along x is $-2\delta\omega_o x$ (and similarly along y). The standard averaging (two-time scale) procedure would yield slow evolution equations

$$\dot{a} \equiv \frac{da}{d\tau} = -\alpha a - \cos\theta + \frac{3}{8}\,ab^2\sin 2(\psi-\theta) \tag{6a}$$

$$\dot{\theta} = \nu + \frac{\sin\theta}{a} - \frac{a^2}{8} + \frac{b^2}{4}\left(1-\frac{3}{2}\cos 2(\psi-\theta)\right) \tag{6b}$$

$$\dot{b} = -\alpha b + \frac{3}{8}\,ba^2\sin 2(\theta-\psi) \tag{6c}$$

$$\dot{\psi} = \nu - \frac{b^2}{8} + \frac{a^2}{4}\left(1-\frac{3}{2}\cos 2(\theta-\psi)\right) \tag{6d}$$

(In Ref. 9, alternative equations for $a\cos\theta$, $a\sin\theta$, etc. were used). Note that all four variables appear on the right-hand side of system (6), and that the flow divergence is negative everywhere:

$$\frac{1}{a}\frac{\partial}{\partial a}(a\dot{a}) + \frac{1}{a}\frac{\partial}{\partial\theta}(a\dot{\theta}) + \frac{1}{b}\frac{\partial}{\partial b}(b\dot{b}) + \frac{1}{b}\frac{\partial}{\partial\psi}(b\dot{\psi}) = -4\alpha < 0.$$

Consider now, however, a driving force $f_x = -\varepsilon g\rho\sin(\omega t + \theta + \sigma)$, coherent with the response; we allow for a constant phase lag σ and a possible slow adjustment of the driving intensity through the factor ρ. We then obtain the previous equations for $\dot{b}$ and $\dot{\psi}$ together with

$$\dot{a} = -\alpha a + \rho\cos\sigma + \frac{3}{8}ab^2\sin 2(\psi-\theta) \tag{6a'}$$

$$\dot{\theta} = \nu + \frac{\rho}{a}\sin\sigma - \frac{a^2}{8} + \frac{b^2}{4}\left(1 - \frac{3}{2}\cos 2(\psi-\theta)\right). \tag{6b'}$$

Note the following features of the new system of equations: *i)* Only three variables, a, b, and $\beta \equiv 2(\psi-\theta)$ enter the right-hand side of the equations, allowing a reduction of phase-space to dimension three. ii) ω is not given a priori so that we may just set $\nu = 0$ ($\omega=\omega_o$); actually ν disappears altogether from the reduced system of equations. iii) If ρ = constant the divergence of the flow is not negative everywhere, a fact caused by the phase-space dependence of the driving (a similar fact occurs in the optimal Censer pumping of Sec. 2). To simplify matters we assume that ρ-adjustment is such that a (-4α) divergence is maintained; this requires that $\rho a = 1$.

Introducing $A \equiv a^2$, $B \equiv b^2$ one gets

$$\dot{A} = -2\alpha A + 2\cos\sigma + \frac{3}{4}AB\sin\beta \tag{7a}$$

$$\dot{B} = -2\alpha B - \frac{3}{4}AB\sin\beta \tag{7b}$$

$$\dot{\beta} = \frac{3}{4}(A-B)(1-\cos\beta) - 2A^{-1}\sin\sigma. \tag{7c}$$

These equations admit one first integral $A+B-\alpha^{-1}\cos\sigma = c\exp(-2\alpha\tau)$, where $c \equiv A(0)+B(0)-\alpha^{-1}\cos\sigma$; note that there would be <u>no</u> first integral for a <u>nonlinear</u> drag. In the following we shall consider $-\pi/2 < \sigma < \pi/2$, A and B remaining thus positive and nonnegative respectively, for both $A(0) > 0$ and $B(0) > 0$. (If $\cos\sigma < 0$, then A might approach zero; since $\rho^2 = 1/A$, an infinite power would be required from the driving agent). We shall first assume $c < 0$, implying $0 < A < \alpha^{-1}\cos\sigma$ throughout.

A 2^{nd}-order non-autonomous system results:

$$\dot{A} = \left(\frac{\cos\sigma}{\alpha} - A\right)\left(\frac{3}{4} A\sin\beta + 2\alpha\right) + \frac{3}{4} Ace^{-2\alpha\tau}\sin\beta$$

$$\dot{\beta} = \left(\frac{3}{2} A - \frac{3\cos\sigma}{4\alpha}\right)(1-\cos\beta) - \frac{2\sin\sigma}{A} - \frac{3}{4} c(1-\cos\beta)e^{-2\alpha\tau}.$$

Although motion in the β-A plane occurs in the same time scale of the last-terms decay above, we may drop these terms in looking for attractors in that plane. Since the state of the system is not changed by adding 2π to β, trajectories of the resulting autonomous equations are best represented on a cylindrical phase space, A lying along its axis, and β circling around from $-\pi$ to π. The loop $A = \alpha^{-1}\cos\sigma$ is a separatrix. Phase space identification of attractors in the parametric α-σ plane is indicated in Fig. 2. We find regions with i) just planar, ii) just non-planar, and iii) coexisting planar and nonplanar attractors.

For $\sigma < 0$, and $\alpha^2 > \frac{3}{16}(1+\text{sen}\,\sigma)$, all trajectories, born at A=0, end in a limit cycle $A = \alpha^{-1}\cos\sigma$, $\dot{\beta} = F(\beta)$; the approach is monotonical if $\alpha^2 > \frac{3}{8}\cos\sigma$ <u>and</u> $\alpha^2 > -3\cos^2\sigma/32\sin\sigma$, non-monotonical otherwise. This long term motion is planar (B=0); its frequency is less than ω_o because $\dot{\theta} = \alpha\tan\sigma - (8\alpha)^{-1}\cos\sigma < 0$.

As $\alpha^2 - \frac{3}{16}(1+\sin\sigma)$ becomes negative, two fixed points are born at some negative β, through a fold bifurcation, saddle on the right, a stable node on the left (for α^2 low enough they change into unstable and stable foci). Thus, for $\alpha^2 < \frac{3}{16}(1+\sin\sigma)$ long term planar and non-planar motions coexist. The catchment or attracting bassins can not be determined, however, from just the limit autonomous system for β and A.

As σ becomes positive, a second fold bifurcation generates one saddle and one stable node within the limit cycle, for positive and negative β respectively (for $\sigma = 0$, a saddle-node lies at $\beta = 0$, which is a separatrix). Note that since β is an irrelevant variable for a planar limit motion, and we still have $\dot{\theta} = \alpha\tan\sigma - (8\alpha)^{-1}\cos\sigma$, there is apparently no change of physical signifance in the long term motion, at this bifurcation.

The fact that there is no limit cycle any more has physical

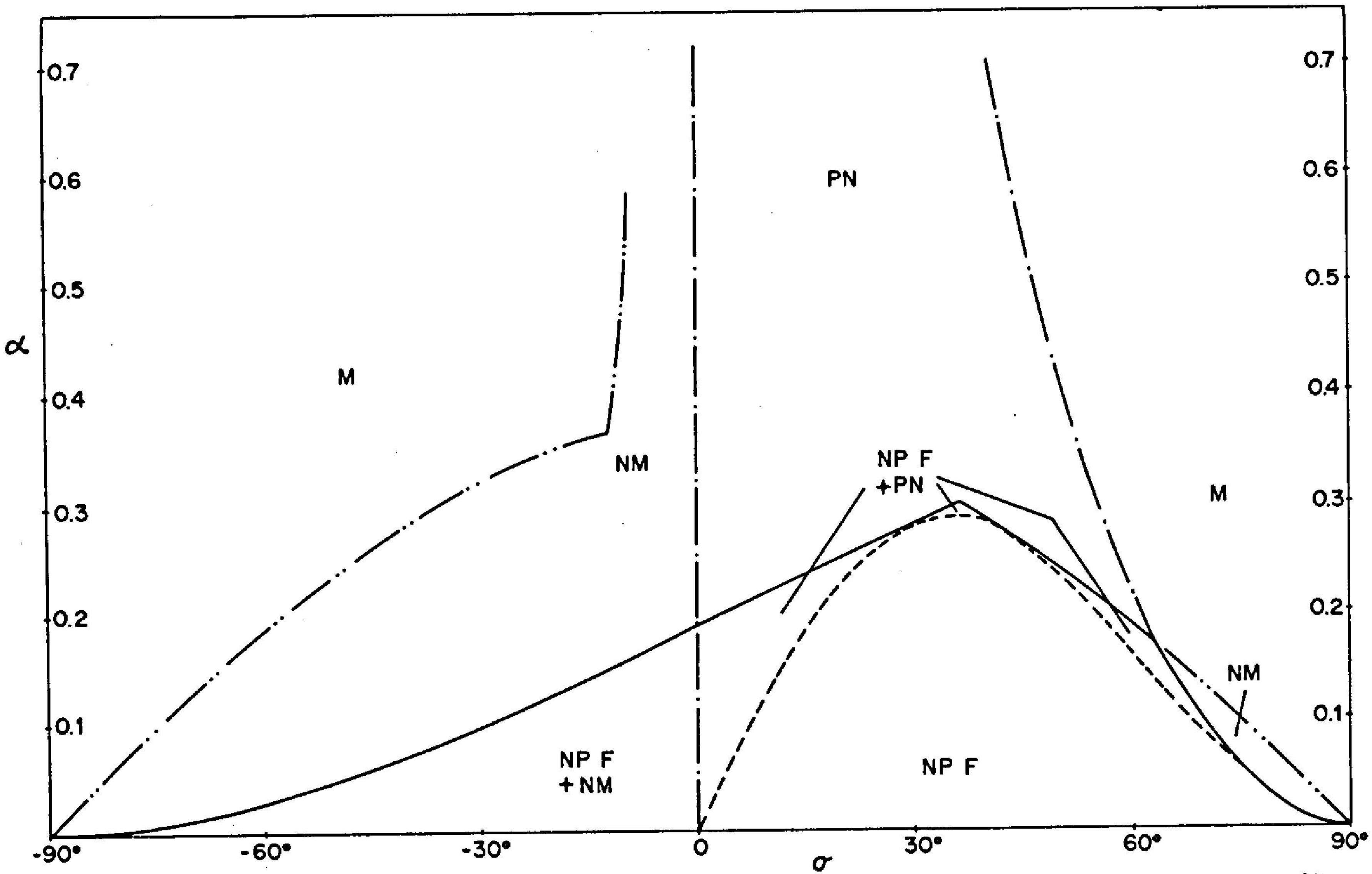

Fig. 2 Attractors of system A-B in the parametric plane α-σ: planar above full line (stable node PN, limit cycle approached monotonically M or nonmonotonically NM); nonplanar below dashed line (stable focus or node NP F); both types in between. There exist transcritical (---) and fold (——, —·—) bifurcations.

consequences however. For $\sigma > 0$ and α^2 less than all three: $\frac{3}{16}(1+\sin\sigma)$, $\frac{3}{8}\cos\sigma$, and $\frac{3}{4}(\sin\sigma)^{-1}\cos^2\sigma$, a (transcritical) second bifurcation takes place: As $\alpha^2 - \frac{3}{4}\sin\sigma\cos^2\sigma$ becomes negative, the nonplanar saddle crosses the planar circle $A = \alpha^{-1}\cos\sigma$, where it collides with the stable node there; afterwards, a stable fixed point emerges in the unphysical region $A > \alpha^{-1}\cos\sigma$, the two planar fixed points now being saddles. Thus, for $\alpha^2 < \frac{3}{4}\sin\sigma\cos^2\sigma$, there is just a nonplanar (point) attractor.

For α^2 larger than, at least, one quantity, $\frac{3}{16}(1+\sin\sigma)$, $\frac{3}{8}\cos\sigma$, or $\frac{3}{4}(\sin\sigma)^{-1}\cos^2\sigma$, there is no nonplanar attractors. As σ increases, $\alpha^2 - \frac{3}{4}(\sin\sigma)^{-1}\cos^2\sigma$ becoming negative, the saddle and stable node at the circle $A = \alpha^{-1}\cos\sigma$ collide in a saddle-node at the point $\beta = \pm\pi$, and afterwards, the circle is again a limit cycle (approached either monotonically or non-monotonically).

The above results will clearly apply to the $c > 0$ case too. For α-σ regions with planar and non-planar attractors, the attracting bassins will be somehow connected with the sign of c.

Chaos was found in Ref. 9 in a weakly-forced, spherical pendulum with linear drag and forcing phase independent of response. The present case of coherent forcing has not led to chaos. We intend to consider a nonlinear drag in the inmediate future.

REFERENCES

1. Neira de Mosquera, A., Semanario Pintoresco Español (Madrid), 338 (1852).
2. Walker, J., The Flying Circus of Physics (Wiley, New York, 1975), p. 38.
3. Berry, M.V., AIP Conference Proceedings 46, 16 (1978); Falk, L., Am. J. Phys. 47, 325 (1979).
4. Sanmartín, J.R., Am. J. Phys. 52, 937 (1984).
5. Flaherty, J.E. and Hoppensteadt, F.C., Stud. Appl. Math. 18, 5 (1978); Holmes, P.J. and Rand, D.A., Q. Appl. Math. 35, 495 (1978); Levi, M., Mem. Am. Math. Soc. 214, 1 (1981); Ueda, Y. and Akamatsu, N., IEEE Trans. Circuits Syst. CAS-28, 217 (1980); Shaw, R., Z. Naturf. 36a, 80 (1981)
6. Holmes, P.J. and Rand, D.A., J. Sound Vibr. 44, 237 (1976); Holmes, P.J. and Whitley, D.C., Physica 7D, 111 (1983); Parlitz, U. and Lauterborn, W., Phys. Lett. 107A, 351 (1985); Räty, R., van Boehm, J.

and Isömaki, H.M., Phys. Lett. 103A, 289 (1984). Ueda, Y., J. Stat. Phys. 20, 181 (1979); in New Approaches to Nonlinear Problems in Dinamics, P.J. Holmes, ed., pp. 311-322 (SIAM, Philadelphia, 1980); in Nonlinear Dynamics, R.G. Helleman, ed., pp. 422-434 (N.Y. Academy of Sciences, New York, 1980).

7. D'Humieres, D., Beasley, M.R., Huberman, B.A. and Libchaber, A., Phys. Rev. A 26, 3483 (1982); Koch, B.P., Leven, R.W., Pompe, B. and Wilke, C., Phys. Lett. 96A, 219 (1983); Mac Donald, A.H. and Plischke, M., Phys. Rev. 27B, 201 (1983); McLaughlin, J.B., J. Stat. Phys. 24, 375 (1981).

8. Strong, C.L., Sci. Am. 210, 136 (1964); Hilton, W.A., Am. J. Phys. 46, 436 (1978); Crane, H.R., Am. J. Phys. 49, 1004 (1981).

9. Miles, J. Physica 11D, 309 (1984); Adv. Appl. Mech. 24, 189 (1984).